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On (m, n)-closed ideals of commutative rings

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Let R be a commutative ring with $1 \neq 0$, and let I be a proper ideal of R. Recall that I is an n-absorbing ideal if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. We define I to be a semi-n-absorbing ideal if $x^{n+1} \in I$ for $x \in R$ implies $x^n \in I$. More generally, for positive integers m and n, we define I to be an (m, n)-closed ideal if $x^m \in I$ for $x \in R$ implies $x^n \in I$. A number of examples and results on (m, n)-closed ideals are discussed in this paper.

Keywords: Prime ideal; radical ideal; 2-absorbing ideal; n-absorbing ideal.

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1. Introduction

Let R be a commutative ring with $1 \neq 0$, I a proper ideal of R, and n a positive integer. As in [1], I is called an *n-absorbing ideal* of R if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. Thus a 1-absorbing ideal is just a prime ideal. In this paper, we define I to be a *semi-n-absorbing ideal* of R if $x^{n+1} \in I$ for $x \in R$ implies $x^n \in I$. Clearly, an n-absorbing ideal is also a semi-n-absorbing ideal, and a semi-1-absorbing ideal is just a radical (semiprime) ideal. Hence n-absorbing (respectively, semi-n-absorbing) ideals generalize prime (respectively, radical) ideals. More generally, for positive integers m and n, we define I to be an *(m, n)-closed ideal* of R if $x^m \in I$ for $x \in R$

implies $x^n \in I$. Thus I is a semi-n-absorbing ideal if and only if I is an $(n+1, n)$-closed ideal, and I is a radical ideal if and only if I is a $(2, 1)$-closed ideal. In fact, an n-absorbing ideal is (m, n)-closed for every positive integer m. Clearly, a proper ideal is (m, n)-closed for $1 \leq m \leq n$; so we often assume that $1 \leq n < m$.

The concept of 2-absorbing ideals was introduced in [6] and then extended to n-absorbing ideals in [1]. Several related concepts, such as 2-absorbing primary ideals, have been studied in [7–10, 16]. Other generalizations of prime ideals are investigated in [3–5, 11].

In Sec. 2, we give the basic properties of semi-n-absorbing ideals and (m, n)-closed ideals. We also determine when every proper ideal of R is (m, n)-closed for integers $1 \leq n < m$. In Sec. 3, we specialize to the case of principal ideals in integral domains. For an integral domain R, we determine $\mathcal{R}(I) = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid I$ is (m, n)-closed$\}$ for $I = p_1^{k_1} \cdots p_i^{k_i} R$, where $p_1, \ldots, p_i$ are nonassociate prime elements of R and $k_1, \ldots, k_i$ are positive integers. In Sec. 4, we continue the study of (m, n)-closed ideals and give several examples to illustrate earlier results. For a proper ideal I of R, we investigate the two functions f_I and g_I defined by $f_I(m) = \min\{n \mid I$ is (m, n)-closed$\}$ and $g_I(n) = \sup\{m \mid I$ is (m, n)-closed$\}$.

We assume throughout that all rings are commutative with $1 \neq 0$ and that $f(1) = 1$ for all ring homomorphisms $f : R \to T$. For such a ring R, $\dim(R)$ denotes the Krull dimension of R, $\sqrt{I}$ denotes the radical of an ideal I of R, and $\mathrm{nil}(R)$, $Z(R)$, and $U(R)$ denote the set of nilpotent elements, zero-divisors, and units of R, respectively; and R is *reduced* if $\mathrm{nil}(R) = \{0\}$. Recall that R is *von Neumann regular* if for every $x \in R$, there is a $y \in R$ such that $x^2 y = x$, and that R is π*-regular* if for every $x \in R$, there are $y \in R$ and a positive integer n such that $x^{2n} y = x^n$. Moreover, R is π-regular (respectively, von Neumann regular) if and only if $\dim(R) = 0$ (respectively, R is reduced and $\dim(R) = 0$) ([13, Theorem 3.1, p. 10]). Thus R is π-regular if and only if $R/\mathrm{nil}(R)$ is von Neumann regular. As usual, $\mathbb{N}, \mathbb{Z}, \mathbb{Z}_n$, and $\mathbb{Q}$ will denote the positive integers, integers, integers modulo n, and rational numbers, respectively. For any undefined concepts or terminology, see [12, 13], or [14].

2. Properties of (m, n)-Closed Ideals

We start with the following observations and examples. Recall that if $M_1, \ldots, M_n$ are maximal ideals of R, then $M_1 \cdots M_n$ is an n-absorbing ideal of R ([1, Theorem 2.9]); an analogous result holds for semi-n-absorbing ideals.

Theorem 2.1. *Let R be a commutative ring.*

(1) *A radical ideal of R is (m, n)-closed for all positive integers m and n.*
(2) *An n-absorbing ideal of R is a semi-n-absorbing ideal (i.e. $(n+1, n)$-closed ideal) of R for every positive integer n.*
(3) *An (m, n)-closed ideal of R is (m', n')-closed for all positive integers $m' \leq m$ and $n' \geq n$.*

(4) *An n-absorbing ideal of R is (m, n)-closed for every positive integer m.*

(5) *Let $P_1, \ldots, P_k$ be radical ideals of R. Then $P_1 \cdots P_k$ is (m, n)-closed for all integers $m \geq 1$ and $n \geq \min\{m, k\}$. In particular, $P_1 \cdots P_k$ is a semi-k-absorbing ideal (i.e. $(k+1, k)$-closed ideal) of R.*

Proof. (1)–(3) follow directly from the definitions.

(4) Let I be an n-absorbing ideal of R for n a positive integer. Suppose that $x^m \in I$ for $x \in R$ and $m > n$ an integer. Then $x^n \in R$ by [1, Theorem 2.1(a)]; so I is (m, n)-closed for $m > n$. Clearly, I is (m, n)-closed for every integer $1 \leq m \leq n$; so I is (m, n)-closed for every positive integer m.

(5) Let $x^m \in P_1 \cdots P_k$ for $x \in R$. Then $x^m \in P_i$ for every $1 \leq i \leq k$, and thus $x \in P_i$ since P_i is a radical ideal of R. Hence $x^k \in P_1 \cdots P_k$; so $x^n \in P_1 \cdots P_k$ for $n \geq \min\{m, k\}$. □

The following examples show that for every integer $n \geq 2$, there is a semi-n-absorbing ideal (i.e. $(n+1, n)$-closed ideal) that is neither a radical ideal nor an n-absorbing ideal, and that there is an ideal that is not a semi-n-absorbing ideal (i.e. $(n+1, n)$-closed ideal) for any positive integer n.

Example 2.2. (a) Let $R = \mathbb{Z}$, $n \geq 2$ an integer, and $I = 2 \cdot 3^n\mathbb{Z}$. Then I is a semi-n-absorbing ideal (i.e. $(n+1, n)$-closed ideal) of R by Theorem 2.1(5) (let $P_1 = 6\mathbb{Z}$ and $P_2 = \cdots = P_n = 3\mathbb{Z}$). In fact, I is a semi-m-absorbing ideal for every integer $m \geq n$. However, $(2 \cdot 3^{n-1})^2 \in I$ and $2 \cdot 3^{n-1} \notin I$; so I is not a radical ideal of R. Moreover, $2 \cdot 3^n \in I$, $3^n \notin I$, and $2 \cdot 3^{n-1} \notin I$; so I is not an n-absorbing ideal of R (but, I is an $(n+1)$-absorbing ideal of R). Note that for $n = 1$, $I = 6\mathbb{Z}$ is a semi-1-absorbing ideal (i.e. radical ideal) of R, but not a 1-absorbing ideal (i.e. prime ideal) of R.

(b) Let $R = \mathbb{Q}[\{X_n\}_{n \in \mathbb{N}}]$ and $I = (\{X_n^n\}_{n \in \mathbb{N}})$. Then $X_{n+1}^{n+1} \in I$ and $X_{n+1}^n \notin I$ for every positive integer n; so I is not a semi-n-absorbing ideal (i.e. $(n+1, n)$-closed ideal) for any positive integer n. Thus I is (m, n)-closed if and only if $1 \leq m \leq n$.

(c) Let R be a commutative Noetherian ring. Then every proper ideal of R is an n-absorbing ideal of R, and hence a semi-n-absorbing ideal of R, for some positive integer n ([1, Theorem 5.3]). Thus, by Theorem 2.1(4), for every proper ideal I of R, there is a positive integer n such that I is (m, n)-closed for every positive integer m. Note that the ring in (b) is not Noetherian.

(d) Clearly, an n-absorbing ideal of R is also an $(n+1)$-absorbing ideal of R. However, this need not be true for semi-n-absorbing ideals. For example, it is easily seen that $I = 16\mathbb{Z}$ is a semi-2-absorbing ideal (i.e. $(3, 2)$-closed ideal) of $\mathbb{Z}$, but not a semi-3-absorbing ideal (i.e. $(4, 3)$-closed ideal) of $\mathbb{Z}$.

(e) Let R be a valuation domain. Then a radical ideal of R is also a prime ideal of R ([12, Theorem 17.1]), i.e. a semi-1-absorbing ideal of R is a 1-absorbing ideal of R. However, a semi-n-absorbing ideal of R need not be an n-absorbing ideal of R for $n \geq 2$. For example, let $R = \mathbb{Z}_{(2)}$ and $I = 16\mathbb{Z}_{(2)}$. Then R is a DVR, and it is easily verified that I is a semi-2-absorbing ideal (i.e. $(3, 2)$-closed ideal) of R, but not a 2-absorbing ideal of R.

In general, a product of (m, n)-closed ideals need not be (m, n)-closed (e.g. a product of radical ideals need not be a radical ideal). The next result generalizes Theorem 2.1(5) (also, see Theorem 4.1(9)).

Theorem 2.3. *Let R be a commutative ring, $m_1, \ldots, m_k, n_1, \ldots, n_k$ positive integers, and $I_1, \ldots, I_k$ be ideals of R such that I_i is (m_i, n_i)-closed for $1 \leq i \leq k$.*

(1) *$I_1 \cap \cdots \cap I_k$ is (m, n)-closed for all positive integers $m \leq \min\{m_1, \ldots, m_k\}$ and $n \geq \min\{m, \max\{n_1, \ldots, n_k\}\}$.*
(2) *$I_1 \cdots I_k$ is (m, n)-closed for all positive integers $m \leq \min\{m_1, \ldots, m_k\}$ and $n \geq \min\{m, n_1 + \cdots + n_k\}$.*

Proof. (1) Let $x^m \in I_1 \cap \cdots \cap I_k$ for $x \in R$, $m \leq \min\{m_1, \ldots, m_k\}$, and $1 \leq i \leq k$. Then $x^m \in I_i$, and thus $x^{m_i} \in I_i$; so $x^{n_i} \in I_i$ since I_i is (m_i, n_i)-closed. Hence $x^n \in I_1 \cap \cdots \cap I_k$ for $n \geq \max\{n_1, \ldots, n_k\}$. Thus $x^n \in I_1 \cap \cdots \cap I_k$ for $n \geq \min\{m, \max\{n_1, \ldots, n_k\}\}$.

(2) Let $x^m \in I_1 \cdots I_k$ for $x \in R$, $m \leq \min\{m_1, \ldots, m_k\}$, and $1 \leq i \leq k$. Then $x^m \in I_i$, and thus $x^{m_i} \in I_i$; so $x^{n_i} \in I_i$ since I_i is (m_i, n_i)-closed. Hence $x^{n_1 + \cdots + n_k} \in I_1 \cdots I_k$; so $x^n \in I_1 \cdots I_k$ for $n \geq n_1 + \cdots + n_k$. Thus $x^n \in I_1 \cdots I_k$ for $n \geq \min\{m, n_1 + \cdots + n_k\}$. □

Recall that two ideals I and J of a commutative ring R are *comaximal* if $I + J = R$, and in this case, $IJ = I \cap J$.

Corollary 2.4. *Let R be a commutative ring, m and n positive integers, and $I_1, \ldots, I_k$ be (m, n)-closed ideals (respectively, semi-n-absorbing ideals) of R.*

(1) *$I_1 \cap \cdots \cap I_k$ is an (m, n)-closed ideal (respectively, semi-n-absorbing ideal) of R.*
(2) *If $I_1, \ldots, I_k$ are pairwise comaximal, then $I_1 \cdots I_k$ is an (m, n)-closed ideal (respectively, semi-n-absorbing ideal) of R.*

Let m and n be positive integers. In [1], we defined a proper ideal I of a commutative ring R to be a *strongly n-absorbing ideal* of R if whenever $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R, then there are n of the I_i's whose product is in I. Clearly, a strongly n-absorbing ideal is also an n-absorbing ideal, and in [1], we gave several cases where the two concepts are equivalent and conjectured that they are always equivalent. Analogously, we define a proper ideal I of R to be a *strongly semi-n-absorbing ideal* of R if $J^n \subseteq I$ whenever $J^{n+1} \subseteq I$ for an ideal J of R, and more generally, we say that a proper ideal I of R is a *strongly (m, n)-closed ideal* of R if $J^n \subseteq I$ whenever $J^m \subseteq I$ for an ideal J of R. Clearly, every proper ideal of R is strongly (m, n)-closed for $1 \leq m \leq n$, a strongly (m, n)-closed ideal of R is an (m,n)-closed ideal of R, and an $(m,1)$-closed ideal of R is also strongly $(m,1)$-closed. However, an (m,n)-closed ideal of R need not be a strongly (m,n)-closed ideal of R; we have the following example.

Example 2.5. Let $R = \mathbb{Z}[X,Y]$, $I = (X^2, 2XY, Y^2)$, and $J = \sqrt{I} = (X,Y)$. Suppose that $a^m \in I$ for $a \in R$ and m a positive integer. Then $a \in \sqrt{I}$, and thus $a = bX + cY$ for some $b, c \in R$. Hence $a^2 = (bX+cY)^2 = b^2X^2 + 2bcXY + c^2Y^2 \in I$, and thus I is an $(m,2)$-closed ideal of R for every positive integer m. It is easily checked that $J^m \subseteq I$ for every integer $m \geq 3$. However, $J^2 \not\subseteq I$ since $XY \notin I$; so I is not a strongly $(m,2)$-closed ideal of R for any integer $m \geq 3$.

In view of Example 2.5, we have the following result.

Theorem 2.6. *Let R be a commutative ring, m a positive integer, I an $(m,2)$-closed ideal of R, and J an ideal of R.*

(1) *If $J^m \subseteq I$, then $2J^2 \subseteq I$.*
(2) *Suppose that $2 \in U(R)$. If $J^m \subseteq I$, then $J^2 \subseteq I$ (i.e. I is a strongly $(m,2)$-closed ideal of R).*

Proof. (1) Let $x, y \in J$. Then $x^m, y^m, (x+y)^m \in I$ since $J^m \subseteq I$, and thus $x^2, y^2, (x+y)^2 \in I$ since I is $(m,2)$-closed. Hence $2xy = (x+y)^2 - x^2 - y^2 \in I$, and thus $2J^2 \subseteq I$.

(2) This follows directly from (1). □

Let I be an (m,n)-closed ideal of a commutative ring R. By Example 2.5, it is possible that $x^n \in I$ for every $x \in J = \sqrt{I}$, but $J^n \not\subseteq I$. It is also possible that $x^n \in I$ for every $x \in J = \sqrt{I}$, but $J^m \not\subseteq I$. Finally, it is possible to have $x^m \notin I$ for some $x \in \sqrt{I}$. We have the following examples.

Example 2.7. (a) Let $R = \mathbb{Z}_2[X,Y,Z]$, $I = (X^2, Y^2, Z^2)$, and $J = \sqrt{I} = (X,Y,Z)$. Let $a \in J$. Then $a = bX + cY + dZ$ for some $b, c, d \in R$. Thus $a^2 = b^2X^2 + c^2Y^2 + d^2Z^2 \in I$; so I is a $(3,2)$-closed ideal of R. However, $J^3 \not\subseteq I$ since $XYZ \notin I$.

(b) Let $R = \mathbb{Z}$ and $I = 16\mathbb{Z}$. Then I is a $(3,2)$-closed ideal of R. However, $2 \in \sqrt{I} = 2\mathbb{Z}$, but $2^3 = 8 \notin I$.

The next theorem is the (m,n)-closed analog for well-known localization results about prime, radical, and n-absorbing ideals ([1, Theorem 4.1]).

Theorem 2.8. *Let R be a commutative ring, m and n positive integers, I an (m,n)-closed ideal of R, and S a multiplicatively closed subset of R such that $I \cap S = \emptyset$.*

(1) *I_S is an (m,n)-closed ideal of R_S. In particular, if I is a semi-n-absorbing ideal of R, then I_S is a semi-n-absorbing ideal of R_S.*

(2) *If* $n = 2$, $2 \in S$, *and* $J^m \subseteq I_S$ *for an ideal* J *of* R_S, *then* $J^2 \subseteq I_S$ (*i.e.* I_S *is a strongly* $(m, 2)$*-closed ideal of* R_S).

Proof. (1) Let $x^m \in I_S$ for $x \in R_S$. Then $x = r/s$ for some $r \in R$ and $s \in S$, and thus $x^m = r^m/s^m = i/t$ for some $i \in I$ and $t \in S$. Hence $r^m tz = s^m iz \in I$ for some $z \in S$, and thus $(rtz)^m \in I$. Hence $(rtz)^n \in I$ since I is (m, n)-closed, and thus $x^n = r^n/s^n = r^n t^n z^n/s^n t^n z^n \in I_S$. Hence I_S is an (m, n)-closed ideal of R_S. The "in particular" statement is clear.

(2) Suppose that $J^m \subseteq I_S$ for an ideal J of R_S. Then $2 \in U(R_S)$ since $2 \in S$, and thus $J^2 \subseteq I_S$ by Theorem 2.6(2). □

Corollary 2.9. *Let* R *be a commutative ring*, I *a proper ideal of* R, *and* m *and* n *positive integers. Then* I *is an* (m, n)*-closed ideal of* R *if and only if* I_P *is an* (m, n)*-closed ideal of* R_P *for every prime* (*or maximal*) *ideal of* R *containing* I. *In particular*, I *is a semi-n-absorbing ideal if and only if* I *is locally a semi-n-absorbing ideal.*

Proof. (⇒) This follows directly from Theorem 2.8(1).

(⇐) Let $x^m \in I$ for $x \in R$, $J = \{r \in R \,|\, rx^n \in I\}$ (an ideal of R), and P be a prime ideal of R with $I \subseteq P$. Then $(x/1)^m \in I_P$; so $(x/1)^n \in I_P$ since I_P is (m, n)-closed. Thus $sx^n \in I$ for some $s \in R \backslash P$; so $J \nsubseteq P$. Clearly, $J \nsubseteq Q$ for every prime ideal Q of R with $I \nsubseteq Q$. Hence $J = R$; so $x^n \in I$. Thus I is (m, n)-closed.

The "in particular" statement is clear. □

The next theorem and corollary extend well-known results about prime, radical, and n-absorbing ideals ([1, Theorem 4.2, Corollary 4.3]) to (m, n)-closed ideals; their proofs are left to the reader.

Theorem 2.10. *Let* R *and* T *be commutative rings*, m *and* n *positive integers*, *and* $f : R \to T$ *a homomorphism.*

(1) *If* J *is an* (m, n)*-closed ideal* (*respectively*, *semi-n-absorbing ideal*) *of* T, *then* $f^{-1}(J)$ *is an* (m, n)*-closed ideal* (*respectively*, *semi-n-absorbing ideal*) *of* R.
(2) *If* f *is surjective and* I *is an* (m, n)*-closed ideal* (*respectively*, *semi-n-absorbing ideal*) *of* R *containing* $\ker f$, *then* $f(I)$ *is an* (m, n)*-closed ideal* (*respectively*, *semi-n-absorbing ideal*) *of* T.

Corollary 2.11. *Let* m *and* n *be positive integers.*

(1) *Let* $R \subseteq T$ *be an extension of commutative rings. If* J *is an* (m, n)*-closed ideal* (*respectively*, *semi-n-absorbing ideal*) *of* T, *then* $J \cap R$ *is an* (m, n)*-closed ideal* (*respectively*, *semi-n-absorbing ideal*) *of* R.
(2) *Let* $I \subseteq J$ *be proper ideals of a commutative ring* R. *Then* J/I *is an* (m, n)*-closed ideal* (*respectively*, *semi-n-absorbing ideal*) *of* R/I *if and only if* J *is an* (m, n)*-closed ideal* (*respectively*, *semi-n-absorbing ideal*) *of* R.

Recall that an ideal of $R \times S$ has the form $I \times J$ for I an ideal of R and J an ideal of S. For a ring T, it will be convenient to define the improper ideal T to be an $(\infty, 1)$-closed ideal of T; then the following theorem holds for all ideals of $R \times S$ (also, see Theorem 4.1(9) and Remark 4.2(d)). The n-absorbing ideal analog of the next theorem was given in [1, Theorem 4.7]; its proof is also left to the reader.

Theorem 2.12. *Let R and S be commutative rings, I an (m_1, n_1)-closed ideal of R, and J an (m_2, n_2)-closed ideal of S. Then $I \times J$ is an (m, n)-closed ideal of $R \times S$ for all positive integers $m \leq \min\{m_1, m_2\}$ and $n \geq \max\{n_1, n_2\}$.*

It is well-known that every proper ideal of a commutative ring R is a prime ideal if and only if R is a field (this is the very first exercise in [14]), and it is easily shown that every proper ideal of R is a radical ideal if and only if R is von Neumann regular. Our next goal is to determine when every proper ideal of R is (m, n)-closed. The following result is included for further reference.

Theorem 2.13. *Let R be a commutative ring and n a positive integer.*

(1) *Every proper ideal of R is a prime ideal if and only if R is a field.*
(2) *Every proper ideal of R is a radical ideal if and only if R is von Neumann regular.*
(3) *If every proper ideal of R is an n-absorbing ideal, then $\dim(R) = 0$ and R has at most n maximal ideals.*

Proof. (1) This result is well known ([14, Exercise 1, p. 7]).

(2) First, suppose that every proper ideal of R is a radical ideal. Let $x \in R$ be a nonunit. Then x^2R is a radical ideal, and thus $x \in x^2R$; so $x = x^2y$ for some $y \in R$. If $x \in U(R)$, then $x = x^2x^{-1}$ with $x^{-1} \in R$. Hence R is von Neumann regular.

Conversely, suppose that R is von Neumann regular. Let I be a proper ideal of R and $x^2 \in I$ for $x \in R$. Then $x = x^2y$ for some $y \in R$, and thus $x = x^2y \in I$. Hence I is a radical ideal.

(3) This is [1, Theorem 5.9]. □

Theorem 2.14. *Let R be a commutative ring and m and n integers with $1 \leq n < m$. Then the following statements are equivalent.*

(1) *Every proper ideal of R is an (m, n)-closed ideal of R.*
(2) $\dim(R) = 0$ *and $w^n = 0$ for every $w \in \mathrm{nil}(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $w \in \mathrm{nil}(R)$. Then w^mR is an (m, n)-closed ideal of R; so $w^n \in w^mR$ since $w^m \in w^mR$. Thus $w^n = w^mz$ for some $z \in R$. Hence $w^n(1-w^{m-n}z) = 0$, and thus $w^n = 0$ since $1 - w^{m-n}z \in U(R)$ because $w^{m-n}z \in \mathrm{nil}(R)$ since $m > n$. Suppose, by way of contradiction, that $\dim(R) \geq 1$. Then there are prime ideals $P \subsetneq Q$ of R. Let $x \in Q\backslash P$. As above, $x^n \in x^mR$; so $x^n = x^my$ for some $y \in R$. Thus $x^n(1 - x^{m-n}y) = 0 \in P$, and hence $1 - x^{m-n}y \in P \subseteq Q$ since $x \in Q\backslash P$. But then $1 \in Q$ since $x^{m-n}y \in Q$, a contradiction. Thus $\dim(R) = 0$.

(2) $\Rightarrow$ (1) Let I be a proper ideal of R, and assume that $x^m \in I$ for $x \in R$. Then R is π-regular since $\dim(R) = 0$, and thus $x = eu + w$ for some idempotent $e \in R$, $u \in U(R)$, and $w \in \text{nil}(R)$ by [15, Theorem 13]. If $n = 1$, then R is reduced, and thus R is von Neumann regular since $\dim(R) = 0$. In this case, every proper ideal of R is a radical ideal by Theorem 2.13(2), and hence I is $(m, 1)$-closed. Thus we may assume that $n \geq 2$. Let $k \geq n$; so $w^k = 0$. Then $x^k = (eu + w)^k = eu^k + keu^{k-1}w + \cdots + keuw^{k-1} = e(u^k + ku^{k-1}w + \cdots + kuw^{k-1})$. Hence $v_k = u^k + ku^{k-1}w + \cdots + kuw^{k-1} \in U(R)$ since $u \in U(R)$, $w \in \text{nil}(R)$, and $k \geq 2$; and thus $x^k = ev_k$. In particular, $x^m = eh \in I$ with $h \in U(R)$ since $m > n$, and hence $e = h^{-1}x^m \in I$. Thus $x^k = ev_k \in I$ for every integer $k \geq n$. Hence I is (m, n)-closed. □

In light of Theorem 2.14, and the fact that an (m, n)-closed ideal is also (m', n)-closed for every positive integer $m' \leq m$, we have the following results.

Theorem 2.15. *Let R be a commutative ring and n a positive integer. Then the following statements are equivalent.*

(1) *Every proper ideal of R is (m, n)-closed for every positive integer m.*
(2) *There is an integer $m > n$ such that every proper ideal of R is (m, n)-closed.*
(3) *For every proper ideal I of R, there is an integer $m_I > n$ such that I is (m_I, n)-closed.*
(4) *Every proper ideal of R is a semi-n-absorbing ideal (i.e. $(n+1, n)$-closed ideal) of R.*
(5) $\dim(R) = 0$ *and* $w^n = 0$ *for every* $w \in \text{nil}(R)$.

Proof. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (5) follows from Theorem 2.14. Finally, (5) $\Rightarrow$ (1) follows from Theorem 2.14 for $m > n$, and from the fact that every proper ideal is (m, n)-closed for $1 \leq m \leq n$. □

Corollary 2.16. *Let R be a reduced commutative ring. Then the following statements are equivalent.*

(1) *Every proper ideal of R is a radical ideal.*
(2) *Every proper ideal of R is (m, n)-closed for all positive integers m and n.*
(3) *There is a positive integer n such that every proper ideal of R is (m, n)-closed for every integer $m \geq n$.*
(4) *There is a positive integer n such that every proper ideal I of R is (m_I, n)-closed for some integer $m_I > n$.*
(5) *There is a positive integer n such that every proper ideal of R is a semi-n-absorbing ideal (i.e. $(n+1, n)$-closed ideal) of R.*
(6) *R is a von Neumann regular ring.*

Moreover, if R is an integral domain and any of the above conditions hold, then R is a field.

Proof. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5), and (5) $\Rightarrow$ (6) by Theorem 2.14 since a reduced commutative ring R with $\dim(R) = 0$ is von Neumann regular. Also, (6) $\Rightarrow$ (1) by Theorem 2.13(2). The "moreover" statement holds since an integral domain is von Neumann regular if and only if it is a field. □

Corollary 2.17. *Let R be a reduced commutative ring and n a positive integer. Then every proper ideal of R is an n-absorbing ideal of R if and only if R is isomorphic to the direct product of at most n fields.*

Proof. ($\Rightarrow$) R is von Neumann regular by Corollary 2.16 since an n-absorbing ideal is a semi-n-absorbing ideal, and R has at most n maximal ideals by Theorem 2.13(b). Thus R is isomorphic to the direct product of at most n fields by the Chinese Remainder Theorem.

($\Leftarrow$) This follows directly from [1, Corollary 4.8]. □

Remark 2.18. Let R be a commutative Noetherian ring. Then every proper ideal of R is an n-absorbing ideal, and thus a semi-n-absorbing ideal (i.e. $(n+1,n)$-closed ideal) of R, for some positive integer n ([1, Theorem 5.3]). However, if there is a fixed positive integer n such that every proper ideal of R is a semi-n-absorbing ideal of R, then $\dim(R) = 0$ by Theorem 2.15.

3. Principal Ideals

In this section, we determine when the powers of a principal prime ideal of an integral domain are (m,n)-closed. Specifically, let R be an integral domain, $I = p^kR$, where p is a prime element of R and k is a positive integer, and m and n be fixed positive integers with $1 \leq n < m$. We first determine $\mathcal{A}(m,n) = \{k \in \mathbb{N} \mid p^kR$ is (m,n)-closed$\}$. Of course, $\mathcal{A}(m,n) = \mathbb{N}$ for $1 \leq m \leq n$. Later, we fix k, and then determine $\mathcal{R}(p^kR) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid p^kR$ is (m,n)-closed$\}$. Note that these results are independent of the integral domain R and the prime p. Finally, these characterizations are extended to ideals of the form $p_1^{k_1} \cdots p_i^{k_i}R$, where $p_1, \ldots, p_i$ are nonassociate prime elements of R and $k_1, \ldots, k_i$ are positive integers.

Theorem 3.1. *Let R be an integral domain, m and n integers with $1 \leq n < m$, and $I = p^kR$, where p is a prime element of R and k is a positive integer. Then the following statements are equivalent.*

(1) *I is an (m,n)-closed ideal of R.*
(2) *$k = ma+r$, where a and r are integers such that $a \geq 0$, $1 \leq r \leq n$, $a(m \bmod n)+r \leq n$, and if $a \neq 0$, then $m = n+c$ for an integer c with $1 \leq c \leq n-1$.*
(3) *If $m = bn+c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n-1$, then $k \in \{1, \ldots, n\}$. If $m = n+c$ for an integer c with $1 \leq c \leq n-1$, then $k \in \bigcup_{h=1}^{n} \{mi+h \mid i \in \mathbb{Z}$ and $0 \leq ic \leq n-h\}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $I = p^k R$ is an (m, n)-closed ideal of R for integers m and n with $1 \leq n < m$. Then $k = ma + r$, where a and r are integers such that $a \geq 0$ and $0 \leq r \leq m-1$. Assume that $r = 0$; so $a > 0$. Thus $(p^a)^m = p^k \in I$, and hence $(p^a)^n \in I$ since I is (m, n)-closed, which is impossible since $na < ma = k$. Thus $1 \leq r \leq m-1$. Let d be the smallest positive integer such that $(p^d)^m \in I$. Then $m(a+1) = k + m - r > k$ since $r < m$, and $ma < k$ since $r \neq 0$. So $d = a+1$ is the smallest positive integer such that $(p^d)^m \in I$. Then $(p^{a+1})^m \in I$, and hence $(p^{a+1})^n \in I$ since I is (m, n)-closed. Thus $na + n = n(a+1) \geq k = ma + r$. Hence $n \geq a(m-n) + r$ with $a(m-n) \geq 0$; so $1 \leq r \leq n$. Since $n < m$, we have $m = bn + c$ for integers b and c with $b \geq 1$ and $0 \leq c \leq n-1$. Thus $n \geq a(bn + c - n) + r = a(b-1)n + ac + r$. Since $n \geq a(b-1)n + ac + r$ and $ac + r \geq 1$, we have $a(b-1) = 0$, and hence $n \geq ac + r$. Thus $a(m \bmod n) + r \leq n$ since $c = m \bmod n$. Assume that $a \neq 0$. Then $b = 1$ since $a(b-1) = 0$. Hence $m = n + c$ with $1 \leq c \leq n-1$ since $n < m$.

(2) $\Rightarrow$ (1) Suppose that $k = ma + r$, where a and r are integers such that $a \geq 0$, $1 \leq r \leq n$, $a(m \bmod n) + r \leq n$, and if $a \neq 0$, then $m = n + c$ for an integer c with $1 \leq c \leq n-1$. Assume that $x^m \in I$ for $x \in R$. We consider two cases. Case I: Assume that $a = 0$. Then $k = r$, and hence $1 \leq k \leq n$. Then $p \mid x$, and thus $p^k \mid x^k$. Hence $p^k \mid x^n$ since $n \geq k$, and thus $x^n \in I$. Case II: Assume that $a \neq 0$. We show that $p^k \mid x^n$, and hence $x^n \in I$. Then $p \mid x$ and $p^k \mid x^m$ since $x^m \in I$. If $p^k \mid x$, then $x^n \in I$. So assume that $p^k \nmid x$. Let i be the largest positive integer such that $p^i \mid x$. Thus $p^{mi} \mid x^m$ and mi is the largest positive integer such that $p^{mi} \mid x^m$. Hence $mi \geq k$; so $0 \geq k - mi = (ma + r) - mi = m(a - i) + r$. Since $1 \leq r \leq n$, we have $i > a$. Thus $i = a + b$ for an integer $b \geq 1$. Then $k = ma + r$ and $m = n + c$ give $k/n = (ma+r)/n = ((n+c)a+r)/n = (na+ca+r)/n = a+(ca+r)/n \leq a+1$ since $ac + r = a(m \bmod n) + r \leq n$. Since $b \geq 1$, we have $i = a + b \geq a + 1 \geq k/n$, and hence $ni \geq k$. Thus $p^{ni} \mid x^n$ since $p^i \mid x$, and hence $p^k \mid x^n$ since $ni \geq k$. So $x^n \in I$. Thus I is (m, n)-closed.

(2) $\Leftrightarrow$ (3) Note that (3) is just an explicit form of (2). □

Theorem 3.2. *Let R be an integral domain, n a positive integer, and $I = p^k R$, where p is a prime element of R and k is a positive integer. Then the following statements are equivalent.*

(1) *I is a semi-n-absorbing ideal (i.e. $(n+1, n)$-closed ideal) of R.*
(2) *$k = (n+1)a + r$, where a and r are integers such that $a \geq 0$, $1 \leq r \leq n$, and $a + r \leq n$.*
(3) *$k \in \bigcup_{h=1}^{n} \{(n+1)i + h \mid i \in \mathbb{Z}$ and $0 \leq i \leq n - h\}$.*

Moreover, $|\{k \in \mathbb{N} \mid p^k R$ is $(n+1, n)$-closed$\}| = n(n+1)/2$.

Proof. (1) $\Leftrightarrow$ (2) The proof is clear by Theorem 3.1 since an ideal I of R is a semi-n-absorbing ideal if and only if I is $(n+1, n)$-closed.

(2) $\Leftrightarrow$ (3) Note that (3) is just an explicit form of (2).

The "moreover" statement follows from (3). $\square$

Corollary 3.3. *Let R be an integral domain and $I = p^k R$, where p is a prime element of R and k is a positive integer. Then I is a semi-2-absorbing ideal (i.e. $(3, 2)$-closed ideal) of R if and only if $k \in \{1, 2, 4\}$.*

We next extend these results to products of prime powers. We use the well-known fact that if $p_1, \ldots, p_n$ are nonassociate prime elements of an integral domain R, then $p_1^{k_1} R \cap \cdots \cap p_n^{k_n} R = p_1^{k_1} \cdots p_n^{k_n} R$ for all positive integers $k_1, \ldots, k_n$. Note that $p_1^{k_1} \cdots p_n^{k_n} R$ is an m-absorbing ideal of R if and only if $m \geq k_1 + \cdots + k_n$ ([1, Theorem 2.1(d)]).

Theorem 3.4. *Let R be an integral domain, m and n integers with $1 \leq n < m$, and $I = p_1^{k_1} \cdots p_i^{k_i} R$, where $p_1, \ldots, p_i$ are nonassociate prime elements of R and $k_1, \ldots, k_i$ are positive integers. Then the following statements are equivalent.*

(1) *I is an (m, n)-closed ideal of R.*
(2) *$p_j^{k_j} R$ is an (m, n)-closed ideal of R for every $1 \leq j \leq i$.*
(3) *If $m = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n - 1$, then $k_j \in \{1, \ldots, n\}$ for every $1 \leq j \leq i$. If $m = n + c$ for an integer c with $1 \leq c \leq n - 1$, then $k_j \in \bigcup_{h=1}^{n} \{mv + h \mid v \in \mathbb{Z}$ and $0 \leq vc \leq n - h\}$ for every $1 \leq j \leq i$.*

Proof. (1) $\Rightarrow$ (2) Let $I_j = p_j^{k_j} R$. Suppose that $x^m \in I_j$ for $x \in R$. Let $y = x(p_1^{k_1} \cdots p_i^{k_i})/p_j^{k_j} \in R$. Then $y^m \in I$, and hence $y^n \in I$ since I is (m, n)-closed. By construction, $y^n \in I$ if and only if $x^n \in I_j$. Thus I_j is an (m, n)-closed ideal of R for every $1 \leq j \leq i$.

(2) $\Rightarrow$ (1) This is clear by Corollary 2.4(1) since $p_1^{k_1} R \cap \cdots \cap p_i^{k_i} R = p_1^{k_1} \cdots p_i^{k_i} R$.

(2) $\Leftrightarrow$ (3) This is clear by Theorem 3.1. $\square$

Corollary 3.5. *Let R be a principal ideal domain, I a proper ideal of R, and m and n integers with $1 \leq n < m$. Then the following statements are equivalent.*

(1) *I is an (m, n)-closed ideal of R.*
(2) *$I = p_1^{k_1} \cdots p_i^{k_i} R$, where $p_1, \ldots, p_i$ are nonassociate prime elements of R and $k_1, \ldots, k_i$ are positive integers, and one of the following two conditions holds.*
   (a) *If $m = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n - 1$, then $k_j \in \{1, \ldots, n\}$ for every $1 \leq j \leq i$.*
   (b) *If $m = n + c$ for an integer c with $1 \leq c \leq n - 1$, then $k_j \in \bigcup_{h=1}^{n} \{mv + h \mid v \in \mathbb{Z}$ and $0 \leq vc \leq n - h\}$ for every $1 \leq j \leq i$.*

Corollary 3.6. *Let R be an integral domain, $I = p_1^{k_1} \cdots p_i^{k_i} R$, where $p_1, \ldots, p_i$ are nonassociate prime elements of R and $k_1, \ldots, k_i$ are positive integers, and n a* *positive integer. Then the following statements are equivalent.*

(1) *I is a semi-n-absorbing ideal (i.e.* $(n+1,n)$*-closed ideal) of R.*
(2) $k_j \in \bigcup_{h=1}^{n}\{(n+1)v+h \mid v \in \mathbb{Z}$ *and* $0 \le v \le n-h\}$ *for every* $1 \le j \le i$.

Corollary 3.7. *Let R be a principal ideal domain, I a proper ideal of R, and n a positive integer. Then the following statements are equivalent.*

(1) *I is a semi-n-absorbing ideal (i.e.* $(n+1,n)$*-closed ideal) of R.*
(2) $I = p_1^{k_1}\cdots p_i^{k_i}R$, *where* $p_1,\ldots,p_i$ *are nonassociate prime elements of R and* $k_1,\ldots,k_i$ *are positive integers, and* $k_j \in \bigcup_{h=1}^{n}\{(n+1)v+h \mid v \in \mathbb{Z}$ *and* $0 \le v \le n-h\}$ *for every* $1 \le j \le i$.

The next theorem uses Theorem 3.1 to give an easier criterion to determine when p^kR is (m,n)-closed.

Theorem 3.8. *Let R be an integral domain, m and n integers with* $1 \le n < m$, *and* $I = p^kR$, *where p is a prime element R and k is a positive integer. Then the following statements are equivalent.*

(1) *I is an* (m,n)*-closed ideal of R.*
(2) *Exactly one of the following statements holds.*
  (a) $1 \le k \le n$.
  (b) *There is a positive integer a such that* $k = ma + r = na + d$ *for integers r and d with* $1 \le r, d \le n-1$.
  (c) *There is a positive integer a such that* $k = ma + r = n(a+1)$ *for an integer r with* $1 \le r \le n-1$.

Proof. (1) $\Rightarrow$ (2) Suppose that I is (m,n)-closed. Then by Theorem 3.1, $k = ma+r$, where a and r are integers such that $a \ge 0$, $1 \le r \le n$, $a(m \bmod n)+r \le n$, and if $a \ne 0$, then $m = n+c$ for an integer c with $1 \le c \le n-1$. Thus if $a = 0$, then $1 \le k \le n$. Hence assume that $a \ne 0$. Note that $m \bmod n = c$. Since $c \ne 0$ and $ac + r \le n$, we conclude that $1 \le r < n$. Since $k = ma + r$ and $m = n + c$, we have $k = (n + c)a + r = na + ac + r$. Let $d = ac + r$. Then $d \le n$. If $d < n$, then $k = ma + r = na + d$, where $1 \le r, d \le n-1$. If $d = n$, then $k = ma + r = n(a+1)$, where $1 \le r \le n-1$.

(2) $\Rightarrow$ (1) First, suppose that $1 \le k \le n$. Then it is clear that I is an (m,n)-closed ideal of R. Next, suppose that there is an integer $a \ge 1$ such that $k = ma+r = na+d$, where $1 \le r, d \le n-1$. Then $m = n+(d-r)/a$, and thus $m = n+c$ for an integer c with $1 \le c \le n-1$. Hence I is (m,n)-closed by Theorem 3.1. Finally, suppose that there is an integer $a \ge 1$ such that $k = ma + r = n(a+1)$, where $1 \le r \le n-1$. Then $m = n+(n-r)/a = n+c$ for an integer c with $1 \le c \le n-1$, and thus I is (m,n)-closed by Theorem 3.1. □

We next calculate $\mathcal{R}(p^kR) = \{(m,n) \in \mathbb{N}\times\mathbb{N} \mid p^kR \text{ is } (m,n)\text{-closed}\}$ for a fixed positive integer k. The following lemma will be needed.

Lemma 3.9. *Let a, d, m, n, r, and w be positive integers such that $1 \leq r < m$, $1 \leq w < n < m$, and $1 \leq d \leq a$.*

(1) *If $ma + r = na + w$, then $1 \leq r < w < n$ and $1 \leq a < n$.*
(2) *If $ma + r = n(a + 1)$, then $1 \leq r < n$ and $1 \leq a < n$.*
(3) *If $ma + r = n(a + 1) + d$, then either $m = n + 1$ or $1 \leq a < n$.*

Proof. (1) Suppose that $ma+r = na+w$. Then $w-r = a(m-n) > 0$ and $1 \leq w < n$. Thus $1 \leq r < w < n$, and hence $0 < w - r < n$. Thus $a = (w - r)/(m - n) < n$ since $0 < w - r < n$ and $m - n \geq 1$.

(2) Suppose that $ma+r = n(a+1)$. Then $n-r = a(m-n) > 0$. Thus $1 \leq r < n$, and $a = (n - r)/(m - n) < n$ since $0 < n - r < n$ and $m - n \geq 1$.

(3) Suppose that $ma + r = n(a + 1) + d$ and $a \geq n$. Then $0 < m - n = a(m-n)/a = (n+d-r)/a = n/a+d/a-r/a < 2$ since $1 < n \leq a$, $1 \leq d \leq a$, and $r > 0$. Thus $m - n = 1$; so $m = n + 1$. □

For fixed positive integers m and k, we next determine the smallest positive integer n such that $I = p^k R$ is (m,n)-closed. Note that $n \leq m$ since every proper ideal is (m,m)-closed and that I is (m,n')-closed for all positive integers $n' \geq n$. So this determines $\mathcal{R}(p^k R)$. Also, if $m > 1$, then $n = 1$ if and only if $k = 1$, i.e. if and only if I is a prime ideal of R. As usual, $\lfloor x \rfloor$ is the greatest integer, or floor, function.

Theorem 3.10. *Let R be an integral domain and $I = p^k R$, where p is a prime element of R and k is a positive integer. Let m be a positive integer and n be the smallest positive integer such that I is (m,n)-closed.*

(1) *If $m \geq k$, then $n = k$.*
(2) *Let $m < k$ and write $k = ma + r$, where a is a positive integer and $0 \leq r < m$.*
    (a) *If $r = 0$, then $n = m$.*
    (b) *If $r \neq 0$ and $a \geq m$, then $n = m$.*
    (c) *If $r \neq 0$, $a < m$, and $(a + 1) \mid k$, then $n = k/(a + 1)$.*
    (d) *If $r \neq 0$, $a < m$, and $(a + 1) \nmid k$, then $n = \lfloor k/(a + 1) \rfloor + 1$.*

Proof. (1) If $m \geq k$, then $p^m \in I$ implies $p^n \in I$; so $n \geq k$. Clearly, I is (m,k)-closed; so $n = k$ is the smallest positive integer such that I is (m,n)-closed when $m \geq k$.

(2) We may assume that $m > 1$, and $n \leq m$ by the above comments.
    (a) Suppose that $r = 0$. Then I is not $(m, m - 1)$-closed since $(p^a)^m = p^k \in I$ and $(p^a)^{m-1} = p^{ma-a} = p^{k-a} \notin I$. Thus $n = m$ since I is (m,m)-closed.
    (b) Suppose that $r \neq 0$ and $a \geq m$. If $n \neq m$, then $n < m < k$. Thus either $k = ma + r = na + d$ or $k = ma + r = n(a + 1)$, where $1 \leq r, d < n$, by Theorem 3.8. Hence $a < n < m$ by Lemma 3.9(1)(2), a contradiction. Thus $n = m$.

(c) Suppose that $r \neq 0$, $a < m$, and $(a+1) \mid k$. Let $i = k/(a+1)$. Then $k = ma + r = i(a+1)$ with $1 \leq i < m$; so $1 \leq r < i$ by Lemma 3.9(2). By Theorem 3.8, I is an (m, i)-closed ideal and it is clear that i is the smallest such positive integer. Thus $n = i = k/(a+1)$.

(d) Suppose that $r \neq 0$, $a < m$, and $(a+1) \nmid k$. Let $i = \lfloor k/(a+1) \rfloor$. Then $k = ma + r = i(a+1) + d$, where $1 \leq d \leq a$ and $1 \leq i < m$. Thus either $m = i+1$ or $1 \leq d \leq a < i$ by Lemma 3.9(3). First, suppose that $m = i+1$. Since $(a+1) \nmid k$, $k \neq i(a+1)$, and thus I is not (m, i)-closed by Theorem 3.8. Hence $n = m = i+1 = \lfloor k/(a+1) \rfloor + 1$ is the smallest positive integer such that I is (m, n)-closed. Next, suppose that $1 \leq d \leq a < i$ and $m \neq i+1$; so $i+1 < m$. Since $k = i(a+1) + d$, we have $k = (i+1)a + i + d - a$. Let $j = i + d - a \in \mathbb{Z}$. Then $1 \leq j \leq i$ since $1 \leq d \leq a < i$. Thus $\lfloor k/(i+1) \rfloor = a$. Since $k = ma + r = (i+1)a + j$ with $1 \leq j < i+1 < m$, we have $1 \leq r < j \leq i$ by Lemma 3.9(1). Hence I is $(m, i+1)$-closed by Theorem 3.8. Since $(a+1) \nmid k$, we have $k \neq i(a+1)$, and thus I is not (m, i)-closed by Theorem 3.8. Hence $n = i+1 = \lfloor k/(a+1) \rfloor + 1$ is the smallest positive integer such that I is (m, n)-closed. □

For fixed positive integers n and k, we next determine the largest positive integer m (or ∞) such that $I = p^k R$ is (m, n)-closed. (If I is (m, n)-closed for every positive integer m, we will say that I is (∞, n)-closed.) Of course, m can also be found using the previous theorem. Clearly, $m \geq n$ since every proper ideal is (n, n)-closed, and I is (m', n)-closed for every positive integer $m' \leq m$.

Theorem 3.11. *Let R be an integral domain, n a positive integer, and $I = p^k R$, where p is a prime element of R and k is a positive integer.*

(1) *If $n \geq k$, then I is (m, n)-closed for every positive integer m.*
(2) *Let $n < k$ and write $k = na + r$, where a is a positive integer and $0 \leq r < n$. Let m be the largest positive integer such that I is (m, n)-closed.*

(a) *If $a > n$, then $m = n$.*
(b) *If $a = n$ and $r = 0$, then $m = n+1$.*
(c) *If $a = n$ and $r \neq 0$, then $m = n$.*
(d) *If $a < n$, $r = 0$, and $(a-1) \mid k$, then $m = k/(a-1) - 1$.*
(e) *If $a < n$, $r = 0$, and $(a-1) \nmid k$, then $m = \lfloor k/(a-1) \rfloor$.*
(f) *If $a < n$, $r \neq 0$, and $a \mid k$, then $m = k/a - 1$.*
(g) *If $a < n$, $r \neq 0$, and $a \nmid k$, then $m = \lfloor k/a \rfloor$.*

Proof. (1) Let $x^m \in I$ for $x \in R$ and m a positive integer. Then $p \mid x^m$; so $p \mid x$ since p is prime. Thus $p^n \mid x^n$; so $x^n \in I$ since $n \geq k$. Hence I is (m, n)-closed.

(2) By the above comments, $m \geq n$. Suppose that I is (m, n)-closed and $m > n$. If $r = 0$, then $k = m(a-1) + w = na$, where $1 \leq w < n$ and $a - 1 < n$ by Theorem 3.8 and Lemma 3.9(2). If $r \neq 0$, then $k = ma + d = na + r$, where $1 \leq d < r < n$ and $a < n$ by Theorem 3.8 and Lemma 3.9(1).

(a) Suppose that $a > n$. If $m \neq n$, then $m > n$; so either $a - 1 < n$ or $a < n$ by the above comments. In either case, $a \leq n$, a contradiction. Thus $m = n$.

(b) Suppose that $a = n$ and $r = 0$; so $k = n^2$ and $n \geq 2$ since $n < k$. Note that $(p^\alpha)^{n+1} \in I \Rightarrow \alpha(n+1) \geq k = n^2 \Rightarrow \alpha \geq n \Rightarrow \alpha n \geq n^2 = k \Rightarrow (p^\alpha)^n \in I$; so I is $(n+1, n)$-closed. However, I is not $(n+2, n)$-closed since $(p^{n-1})^{n+2} \in I$ and $(p^{n-1})^n \notin I$. Thus $m = n + 1$.

(c) Suppose that $a = n$ and $r \neq 0$. If $m > n$, then $a < n$ by the above comments. This is a contradiction; so $m = n$.

(d) Suppose that $a < n$, $r = 0$, and $(a-1)|k$ (note that $a \geq 2$ since $na = k > n$). Let $f = k/(a - 1)$; so $k = f(a - 1)$ and $a < n < f$. Thus $k = f(a - 1) = (f - 1 + 1)(a - 1) = (f - 1)(a - 1) + a - 1 = na$ with $a - 1 < n$. Hence I is $(f - 1, n)$-closed by Theorem 3.8. Note that I is not (f, n)-closed by Theorem 3.8. Hence $m = f - 1 = k/(a - 1) - 1$ is the largest positive integer such that I is (m, n)-closed.

(e) Suppose that $a < n$, $r = 0$, and $(a - 1)\nmid k$ (as in (d), $a \geq 2$). Let $f = \lfloor k/(a - 1)\rfloor$; so $k = f(a - 1) + d$, where $1 \leq d < a - 1$. Since $a < n < f$, we have $1 \leq d < a - 1 < f$. Since $k = f(a - 1) + d = na$ and $1 \leq d < f$, we have $d < n$ by Lemma 3.9(2). Thus I is (f, n)-closed by Theorem 3.8. Note that by construction of f, if $k = i(a-1)+c$ for some $1 \leq c < a-1$, then $i \leq f$. Thus $m = f = \lfloor k/(a-1)\rfloor$ is the largest positive integer such that I is (m, n)-closed.

(f) Suppose that $a < n$, $r \neq 0$, and $a \,|\, k$. Let $f = k/a$; so $k = fa$ and $f \geq n+1$. Then I is not (f, n)-closed by Theorem 3.8. First, assume that $f - 1 > n$. Thus $k = fa = (f - 1 + 1)a = (f - 1)a + a$. Since $a < n < f - 1$ and $k = (f - 1)a + a = na + r$, we conclude that I is $(f - 1, n)$-closed by Theorem 3.8. Hence, in this case, $m = f - 1 = k/a - 1$ is the largest positive integer such that I is (m, n)-closed. Next, assume that $f - 1 = n$. Then clearly $m = n = k/a - 1$ is again the largest positive integer such that I is (m, n)-closed.

(g) Suppose that $a < n$, $r \neq 0$, and $a \nmid k$. Let $f = \lfloor k/a\rfloor$; so $k = fa + d$, where $1 \leq d < a$. Since $a < n < f$, we have $1 \leq d < a < f$. Since $k = fa + d = na + r$ and $1 \leq d < f$, we have $d < n$ by Lemma 3.9(1). Thus I is (f, n)-closed by Theorem 3.8. Note that by construction of f, if $k = ia + c$ for some $1 \leq c < a$, then $i \leq f$. Thus $m = f = \lfloor k/a\rfloor$ is the largest positive integer such that I is (m, n)-closed. □

The previous two theorems easily extend to products of principal prime ideals. In particular, we can calculate $\mathcal{R}(I) = \{(m, n) \in \mathbb{N} \times \mathbb{N} \,|\, I \text{ is } (m, n)\text{-closed}\}$ for every proper ideal I in a principal ideal domain or every proper principal ideal I in a unique factorization domain.

Theorem 3.12. *Let R be an integral domain and $I = p_1^{k_1} \cdots p_i^{k_i} R$, where $p_1, \ldots, p_i$ are nonassociate prime elements of R and $k_1, \ldots, k_i$ are positive integers.*

(1) *Let m be a positive integer. If n_j is the smallest positive integer such that $p_j^{k_j} R$ is (m, n_j)-closed for $1 \leq j \leq i$, then $n = \max\{n_1, \ldots, n_i\}$ is the smallest positive integer such that I is (m, n)-closed.*

(2) *Let n be a positive integer. If m_j is the largest positive integer (or ∞) such that $p_j^{k_j}R$ is (m_j, n)-closed for $1 \leq j \leq i$, then $m = \min\{m_1, \ldots, m_i\}$ is the largest positive integer (or ∞) such that I is (m, n)-closed.*

Proof. This follows since I is (m, n)-closed if and only if every $p_j^{k_j}R$ is (m, n)-closed by Theorem 3.4. □

4. General Results

Let I be a proper ideal of a commutative ring R. We define $\mathcal{R}(I) = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid I \text{ is } (m, n)\text{-closed}\}$. Thus $\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq m \leq n\} \subseteq \mathcal{R}(I) \subseteq \mathbb{N} \times \mathbb{N}$ and $\mathcal{R}(I) = \mathbb{N} \times \mathbb{N}$ if and only if $\sqrt{I} = I$. We start with some elementary properties of $\mathcal{R}(I)$. If we define $\mathcal{R}(R) = \mathbb{N} \times \mathbb{N}$, then the results in this section hold for all ideals of R.

Theorem 4.1. *Let R be a commutative ring, I and J proper ideals of R, and m, n and k positive integers.*

(1) *$(m, n) \in \mathcal{R}(I)$ for all positive integers m and n with $m \leq n$.*
(2) *If $(m, n) \in \mathcal{R}(I)$, then $(m', n') \in \mathcal{R}(I)$ for all positive integers m' and n' with $1 \leq m' \leq m$ and $n' \geq n$.*
(3) *If $(m, n) \in \mathcal{R}(I)$, then $(km, kn) \in \mathcal{R}(I)$.*
(4) *If $(m, n), (n, k) \in \mathcal{R}(I)$, then $(m, k) \in \mathcal{R}(I)$.*
(5) *If $(m, n), (m + 1, n + 1) \in \mathcal{R}(I)$ for $m \neq n$, then $(m + 1, n) \in \mathcal{R}(I)$.*
(6) *If $(n, 2), (n + 1, 2) \in \mathcal{R}(I)$ for an integer $n \geq 3$, then $(n + 2, 2) \in \mathcal{R}(I)$, and thus $(m, 2) \in \mathcal{R}(I)$ for every positive integer m.*
(7) *If $(m, n) \in \mathcal{R}(I)$ for positive integers m and n with $n \leq m/2$, then $(m+1, n) \in \mathcal{R}(I)$, and thus $(k, n) \in \mathcal{R}(I)$ for every positive integer k.*
(8) *$(m, n) \in \mathcal{R}(I)$ for every positive integer m if and only if $(2n, n) \in \mathcal{R}(I)$.*
(9) *$\mathcal{R}(I \times J) = \mathcal{R}(I) \cap \mathcal{R}(J) \subseteq \mathcal{R}(I \cap J)$.*

Proof. (1)–(4) all follow easily from the definitions.

(5) If $m < n$, then $(m + 1, n) \in \mathcal{R}(I)$ by (1). For $m > n$, suppose that $x^{m+1} \in I$ for $x \in R$. Then $x^{n+1} \in I$ since I is $(m + 1, n + 1)$-closed. Thus $x^m \in I$ since $m \geq n + 1$, and hence $x^n \in I$ since I is (m, n)-closed. Thus I is $(m + 1, n)$-closed.

(6) Suppose that $x^{n+2} \in I$ for $x \in R$. Then $(x^2)^n = x^{2n} \in I$ since $2n \geq n + 2$ because $n \geq 2$. Hence $x^4 = (x^2)^2 \in I$ since I is $(n, 2)$-closed. But then $x^{n+1} \in I$ since $n \geq 3$. Thus $x^2 \in I$ since I is $(n + 1, 2)$-closed. Hence I is $(n + 2, 2)$-closed. Similarly, $(k, 2) \in \mathcal{R}(I)$ for every integer $k \geq n + 3$. So by (2), I is $(k, 2)$-closed for every positive integer k.

(7) Let $x^{m+1} \in I$ for $x \in R$. Then $(x^2)^m = x^{2m} \in I$, and hence $x^{2n} = (x^2)^n \in I$ since I is (m, n)-closed. Thus $x^m \in I$ since $2n \leq m$, and hence $x^n \in I$ since I is (m, n)-closed. Thus I is $(m + 1, n)$-closed. Similarly, $(k, n) \in \mathcal{R}(I)$ for every integer $k \geq n$, and hence $(k, n) \in \mathcal{R}(I)$ for every positive integer k by (2).

(8) This follows directly from (7).

(9) Clearly $I \times J$ is (m,n)-closed if and only if I and J are both (m,n)-closed. Thus $\mathcal{R}(I \times J) = \mathcal{R}(I) \cap \mathcal{R}(J)$. That $\mathcal{R}(I) \cap \mathcal{R}(J) \subseteq \mathcal{R}(I \cap J)$ follows from Corollary 2.4(1). □

Remark 4.2. (a) The $m \neq n$ hypothesis is needed in Theorem 4.1(5) since $(n,n) \in \mathcal{R}(I)$ for every positive integer n.

(b) The $n \geq 3$ hypothesis is needed in Theorem 4.1(6). For $n = 1$, we have $(1,2),(2,2) \in \mathcal{R}(I)$ for every proper ideal I of R, but, in general, $(3,2) \notin \mathcal{R}(I)$. For $n = 2$,we have $(2,2),(3,2) \in \mathcal{R}(I)$ does not imply $(4,2) \in \mathcal{R}(I)$. For example, let $R = \mathbb{Z}$ and $I = 16\mathbb{Z}$. Then $(2,2),(3,2) \in \mathcal{R}(I)$, but $(4,2) \notin \mathcal{R}(I)$.

(c) The inclusion in Theorem 4.1(9) may be strict. For example, let $R = \mathbb{Z}, I = 8\mathbb{Z}$, and $J = 16\mathbb{Z}$. Then $(3,2) \in \mathcal{R}(J) = \mathcal{R}(I \cap J)$. However, $(3,2) \notin \mathcal{R}(I)$; so $\mathcal{R}(I) \cap \mathcal{R}(J) \subsetneq \mathcal{R}(I \cap J)$.

(d) More generally, $\mathcal{R}(I \times J) = \mathcal{R}(I) \cap \mathcal{R}(J)$ for all ideals I and J of commutative rings R and S, respectively.

Let I be a proper ideal of a commutative ring R and m and n positive integers. We define $f_I(m) = \min\{n \mid I \text{ is } (m,n)-\text{closed}\} \in \{1,\ldots,m\}$ and $g_I(n) = \sup\{m \mid I \text{ is } (m,n)-\text{closed}\} \in \{n, n+1,\ldots\} \cup \{\infty\}$; so $f_I : \mathbb{N} \to \mathbb{N}$ and $g_I : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. The columns (respectively, rows) of $\mathcal{R}(I)$ determine f_I (respectively, g_I). Thus either function f_I or g_I is determined by the other, and either function determines $\mathcal{R}(I)$ by Theorem 4.1(2). It is sometimes useful to view f_I (respectively, g_I) as an $\mathbb{N}$-valued (respectively, $\mathbb{N} \cup \{\infty\}$-valued) non-decreasing sequence $f_I = (f_I(m))$ (respectively, $(g_I = (g_I(n)))$. Note that $f_I = (1,1,1,\ldots)$ if and only if $g_I = (\infty,\infty,\infty,\ldots)$, if and only if $\sqrt{I} = I$. If we define $\mathcal{R}(R) = \mathbb{N} \times \mathbb{N}$, then $f_R = (1,1,1,\ldots)$ and $g_R = (\infty,\infty,\infty,\ldots)$. Also, f_I is eventually constant if and only if g_I is eventually constant, if and only if g_I is eventually ∞. We next give some elementary properties of the two functions f_I and g_I.

Theorem 4.3. *Let R be a commutative ring, I a proper ideal of R, and m and n positive integers. Let $f_I(m) = \min\{n \mid I \text{ is } (m,n)\text{-closed}\}$ and $g_I(n) = \sup\{m \mid I \text{ is } (m,n)\text{-closed}\}$.*

(1) $1 \leq f_I(m) \leq m$.
(2) $f_I(m) \leq f_I(m+1)$.
(3) *If $f_I(m) < m$, then either $f_I(m+1) = f_I(m)$ or $f_I(m+1) \geq f_I(m) + 2$.*
(4) $n \leq g_I(n) \leq \infty$.
(5) $g_I(n) \leq g_I(n+1)$.
(6) *If $g_I(n) > n$, then either $g_I(n+1) = g_I(n)$ or $g_I(n+1) \geq g_I(n) + 2$.*

Proof. (1) This is clear since $(n,n) \in \mathcal{R}(I)$ for every positive integer n by Theorem 4.1(1).

(2) This is clear by Theorem 4.1(2).

(3) Suppose that $f_I(m+1) = f_I(m) + 1$. Let $f_I(m) = n$; so $m > n$ and $f_I(m+1) = n+1$. Then $(m, n), (m+1, n+1) \in \mathcal{R}(I)$ and $m > n$; so $(m+1, n) \in \mathcal{R}(I)$ by Theorem 4.1(5). Thus $f_I(m+1) \leq n$, a contradiction.

(4) This is also clear by Theorem 4.1(1).

(5) This is also clear by Theorem 4.1(2).

(6) The proof is similar to that of (3). □

For $f, g : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$, we define $f \leq g$ if and only if $f(n) \leq g(n)$ for every $n \in \mathbb{N}$. Thus $(f \vee g)(n) = \max\{f(n), g(n)\}$ and $(f \wedge g)(n) = \min\{f(n), g(n)\}$ for every $n \in \mathbb{N}$.

Theorem 4.4. *Let R be a commutative ring and I and J proper ideals of R. Let $f_I(m) = \min\{n \,|\, I$ is (m, n)-closed$\}$ and $g_I(n) = \sup\{m \,|\, I$ is (m, n)-closed$\}$. Then the following statements are equivalent.*

(1) $\mathcal{R}(I) \subseteq \mathcal{R}(J)$.
(2) $f_J \leq f_I$, *i.e.* $f_J(m) \leq f_I(m)$ *for every positive integer* m.
(3) $g_I \leq g_J$, *i.e.* $g_I(n) \leq g_J(n)$ *for every positive integer* n.

Proof. It is clear that (1) ⇔ (2) and (2) ⇔ (3). □

The next theorem relates f_I, f_J (respectively, g_I, g_J) and $f_{I \cap J}$ (respectively, $g_{I \cap J}$).

Theorem 4.5. *Let R be a commutative ring and I and J proper ideals of R. Let $f_I(m) = \min\{n \,|\, I$ is (m, n)-closed$\}$ and $g_I(n) = \sup\{m \,|\, I$ is (m, n)-closed$\}$.*

(1) $f_{I \cap J} \leq f_I \vee f_J$.
(2) $g_I \wedge g_J \leq g_{I \cap J}$.
(3) *The following statements are equivalent.*
  (a) $f_{I \cap J} = f_I \vee f_J$
  (b) $g_{I \cap J} = g_I \wedge g_J$.
  (c) $\mathcal{R}(I \cap J) = \mathcal{R}(I) \cap \mathcal{R}(J)$.

Proof. (1) Let $m \in \mathbb{N}, n_1 = f_I(m), n_2 = f_J(m)$, and $n = \max\{n_1, n_2\}$. Then $(m, n) \in \mathcal{R}(I) \cap \mathcal{R}(J) \subseteq \mathcal{R}(I \cap J)$ by Theorem 4.1(2)(9). Thus $f_{I \cap J}(m) \leq n = (f_I \vee f_J)(m)$.

(2) The proof is similar to that of (1).

(3) $(a) \Rightarrow (c)$ Suppose that $f_{I \cap J} = f_I \vee f_J$. Then $f_I, f_J \leq f_{I \cap J}$; so $\mathcal{R}(I \cap J) \subseteq \mathcal{R}(I) \cap \mathcal{R}(J)$ by Theorem 4.4. Thus $\mathcal{R}(I \cap J) = \mathcal{R}(I) \cap \mathcal{R}(J)$ by Theorem 4.1(9).

$(c) \Rightarrow (a)$ Suppose that $\mathcal{R}(I \cap J) = \mathcal{R}(I) \cap \mathcal{R}(J)$. Then $f_I, f_J \leq f_{I \cap J}$ by Theorem 4.4; so $f_I \vee f_J \leq f_{I \cap J}$. Thus $f_{I \cap J} = f_I \vee f_J$ since $f_{I \cap J} \leq f_I \vee f_J$ by (1).

$(b) \Leftrightarrow (c)$ The proof is similar to that of $(a) \Leftrightarrow (c)$. □

The next result gives a case where $\mathcal{R}(I \cap J) = \mathcal{R}(I) \cap \mathcal{R}(J)$; its "moreover" statement generalizes (1) $\Leftrightarrow$ (2) of Theorem 3.4. Recall that two nonunits x and y in an integral domain R are *coprime* if $xR \cap yR = xyR$.

Theorem 4.6. *Let R be an integral domain and $x, y \in R$ coprime elements. Then $\mathcal{R}(xyR) = \mathcal{R}(xR \cap yR) = \mathcal{R}(xR) \cap \mathcal{R}(yR)$. Moreover, $f_{xyR} = f_{xR} \vee f_{yR}$ and $g_{xyR} = g_{xR} \wedge g_{yR}$.*

Proof. By Theorem 4.1(9), we need only to show that $\mathcal{R}(xR \cap yR) \subseteq \mathcal{R}(xR) \cap \mathcal{R}(yR)$. We first show that $\mathcal{R}(xR \cap yR) \subseteq \mathcal{R}(xR)$. Let $(m, n) \in \mathcal{R}(xR \cap yR)$, and suppose that $a^m \in xR$ for $a \in R$. Then $(ay)^m \in xR \cap yR = xyR$, and thus $(ay)^n \in xyR \subseteq xR$ since $xyR = xR \cap yR$ is (m, n)-closed. Hence $(ay)^n \in xR \cap y^n R = xy^n R$ (this follows since x and y are coprime); so $a^n \in xR$. Thus xR is (m, n)-closed; so $(m, n) \in \mathcal{R}(xR)$. Similarly, $(m, n) \in \mathcal{R}(yR)$; so $\mathcal{R}(xR \cap yR) \subseteq \mathcal{R}(xR) \cap \mathcal{R}(yR)$. Hence $\mathcal{R}(xyR) = \mathcal{R}(xR \cap yR) = \mathcal{R}(xR) \cap \mathcal{R}(yR)$. □

The functions f_I and g_I may be strictly increasing (see Example 4.8(d)). However, if R is a commutative Noetherian ring, then f_I and g_I are eventually constant (i.e. g_I is eventually ∞) for every proper ideal I of R (cf. Example 2.2(c)).

Theorem 4.7. *Let R be a commutative ring, n a positive integer, and I an n-absorbing ideal of R. Then $f_I(m) \leq n$ for every positive integer m. Thus f_I and g_I are eventually constant. In particular, if R is Noetherian, then f_I and g_I are eventually constant for every proper ideal I of R.*

Proof. This follows directly from Theorem 2.1(4). The "in particular" statement holds since every proper ideal of a commutative Noetherian ring is an n-absorbing ideal for some positive integer n by [1, Theorem 5.3]. □

Let R be an integral domain and $I = p^k R$, where p is a prime element of R and k is a positive integer. Then Theorem 3.10 computes f_I and Theorem 3.11 computes g_I; the general case for $I = p_1^{k_1} \cdots p_i^{k_i} R$ is given by Theorem 3.12.

We end this section by computing the f_I and g_I functions for several examples.

Example 4.8. (a) Let R be an integral domain and $I = p^{30} R$ for p a prime element of R. By Theorem 3.10, one may easily calculate that $f_I(m) = m$ for $1 \leq m \leq 6$, $f_I(7) = 6, f_I(8) = f_I(9) = 8, f_I(m) = 10$ for $10 \leq m \leq 14, f_I(m) = 15$ for $15 \leq m \leq 29$, and $f_I(m) = 30$ for $m \geq 30$. Using Theorem 3.11 (or the f_I function), one may easily calculate that $g_I(n) = n$ for $1 \leq n \leq 5, g_I(6) = g_I(7) = 7, g_I(8) = g_I(9) = 9, g_I(n) = 14$ for $10 \leq n \leq 14, g_I(n) = 29$ for $15 \leq n \leq 29$, and $g_I(n) = \infty$ for $n \geq 30$.

(b) Let $R = \mathbb{Z}$ and $I = 1260000\mathbb{Z} = 2^5 3^2 5^4 7\mathbb{Z}$. Then $I = I_1 \cap I_2 \cap I_3 \cap I_4$, where $I_1 = 2^5\mathbb{Z}, I_2 = 3^2\mathbb{Z}, I_3 = 5^4\mathbb{Z}$, and $I_4 = 7\mathbb{Z}$. Let $f_i = f_{I_i}$ and $g_i = g_{I_i}$. Then $f_1 = (1, 2, 3, 3, 5, 5, 5, \ldots)$, $f_2 = (1, 2, 2, 2, \ldots)$, $f_3 = (1, 2, 2, 4, 4, 4, \ldots)$,

$f_4 = (1, 1, 1, \ldots)$ and $g_1 = (1, 2, 4, 4, \infty, \infty, \infty, \ldots)$, $g_2 = (1, \infty, \infty, \infty, \ldots)$, $g_3 = (1, 3, 3, \infty, \infty, \infty, \ldots)$, $g_4 = (\infty, \infty, \infty, \ldots)$ by Theorems 3.10 and 3.11, respectively. Thus $f_I = (1, 2, 3, 4, 5, 5, 5, \ldots)$ and $g_I = (1, 2, 3, 4, \infty, \infty, \infty, \ldots)$ by Theorem 3.12.

(c) Let n be a positive integer, p_n be the nth positive prime integer, $R = \mathbb{Z}$, and $I_n = 2^1 3^2 \cdots p_n^n \mathbb{Z}$. Then $f_{I_1} = (1, 1, 1, \ldots)$, $g_{I_1} = (\infty, \infty, \infty, \ldots)$, $f_{I_n} = (1, 2, \ldots, n-1, n, n, n, \ldots)$, and $g_{I_n} = (1, 2, \ldots, n-1, \infty, \infty, \infty, \ldots)$ for $n \geq 2$ by Theorems 3.10–3.12.

(d) Let $R = \mathbb{Q}[\{X_n\}_{n\in\mathbb{N}}]$ and $I = (\{X_n^n\}_{n\in\mathbb{N}})$ as in Example 2.2(b). Then $f_I(m) = g_I(m) = m$ for every positive integer m since I is (m, n)-closed if and only if $1 \leq m \leq n$. Thus $f_I = g_I = (1, 2, 3, \ldots, n-1, n, n+1, \ldots)$.

The final example shows that for P a prime ideal (with $P^4 \subsetneq P^3$) and p a prime element of an integral domain R, the ideals $I = p^4R$ and $J = P^4$ may give distinct functions f_I, f_J and g_I, g_J.

Example 4.9. (a) Let R be an integral domain and $I = p^4R$, where p is a prime element of R. One may easily compute that $f_I(1) = 1, f_I(2) = f_I(3) = 2$, and $f_I(m) = 4$ for $m \geq 4$. Thus $g_I(1) = 1, g_I(2) = g_I(3) = 3$, and $g_I(n) = \infty$ for $n \geq 4$; so $f_I = (1, 2, 2, 4, 4, 4, \ldots)$ and $g_I = (1, 3, 3, \infty, \infty, \infty, \ldots)$.

(b) Let $R = \mathbb{Z}[X] + \sqrt[3]{2}X\mathbb{Z}[\sqrt[3]{2}][X]$. Then $P = (2, X, \sqrt[3]{2}X)$ is a prime ideal of R and $P^4 \subsetneq P^3$. Note that $J = P^4$ is not $(3, 2)$-closed since $(\sqrt[3]{2}X)^3 = 2X^3 \in J$ and $(\sqrt[3]{2}X)^2 \notin J$. Also, $2^4 \in J$ and $2^3 \notin J$; so J is not $(4, 3)$-closed. Clearly, J is $(m, 4)$-closed for every positive integer m. Thus $f_J(m) = m$ for $1 \leq m \leq 3$ and $f_J(m) = 4$ for $m \geq 4$, and hence $g_J(n) = n$ for $1 \leq n \leq 3$ and $g_J(n) = \infty$ for $n \geq 4$; so $f_J = (1, 2, 3, 4, 4, 4, \ldots)$ and $g_J = (1, 2, 3, \infty, \infty, \infty, \ldots)$. Thus $f_I < f_J$ and $g_J < g_I$, where f_I and g_I are from (a).

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